IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Vector solutions of the Laplace equation and the influence of helicity on Aharonov-Bohm scattering

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1994 J. Phys. A: Math. Gen. 27 2143

(http://iopscience.iop.org/0305-4470/27/6/035)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.68
The article was downloaded on 02/06/2010 at 00:26

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 27 (1994) 2143–2160. Printed in the UK

Vector solutions of the Laplace equation and the influence of helicity on Aharonov–Bohm scattering

G N Afanasiev

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow District, 141980, Russia

Received 8 December 1992

Abstract. Vector solutions of the Laplace equation are obtained. Their properties and possible applications are discussed. The multipole toroidal moments appear naturally in this vector basis, removing the mystery of their origin. Conditions are found for the non-radiation of charge and current densities periodically changing with time. Electromagnetic properties of the toroidal solenoid with non-zero helicity, the influence of the latter on Aharonov–Bohm scattering, and an alternative viewpoint on the toroidal solenoid with non-trivial helicity are studied.

1. Introduction

Vector spherical harmonics (VSH) and elementary vector potentials (EVP) [1, 2], closely related to them, are powerful tools for solving radiation [1] and scattering [3] problems occurring in optical [4], particle, nuclear [5] and atomic physics [6]. EVP are the vector solutions of the Helmholtz equation. Much less is known about the vector solutions of the Laplace equation. At first, this seems strange. In fact, as the Helmholtz equation in the long-wavelength limit ($k \to 0$) transforms into the Laplace equation, one may expect the same for its vector solutions. It turns out, however, that contributions of EVP corresponding to electric (E) and longitudinal (L) multipoles diverge in the $k \to 0$ limit. This gave rise to numerous fallacies and controversies in the physical literature, some of which have been discussed recently in the review article [7]. It is the aim of the present consideration to find the correct limiting procedure for the static case.

For the static case, there are known configurations of charges and currents which, being imbedded into the finite region of space S, generate electric and magnetic fields vanishing outside S. The typical representatives are electric capacitors and magnetic solenoids. Is the same situation possible for the time-dependent charge and current densities? Particular examples of that kind were given in [8–10]. In the present consideration, we find general conditions which should satisfy non-radiating charge and particle densities.

A static configuration of the magnetic field enclosed into the finite space region S may be characterized by the number of topological invariants which are unchanged under an arbitrary continuous deformation of S [11]. The simplest one is magnetic flux, which depends on the number of magnetic lines and their total intensity. The next (in complexity) invariant is helicity [12, 13], which measures how much the

0305-4470/94/062143+18 $19.50 © 1994 IOP Publishing Ltd

magnetic lines are coupled with each other. The helicity may be different from zero, even for a single magnetic line, which should be either knotted or internally twisted in this case [14]. Using the superposition of magnetic dipole and toroidal moments we construct a toroidal solenoid (TS) with non-zero helicity and investigate how it affects charge particle scattering. For the helicity to be physically meaningful, it should be a gauge invariant quantity. This requires that the magnetic lines in the vicinity of the boundary enclosing the magnetic flux should be parallel to it [12]. Such a property holds for the solenoids and this in turn justifies the use of helicity for their description.

The plan of our exposition is as follows. The main facts concerning VSH and EVP are collected in section 2. It is shown at the beginning of section 3 that the transition to the $k\to 0$ limit in the solutions of the Helmholtz equation does not always give all the solutions of the Laplace equation. To remove this insufficiency the orthonormal vector basis of the Laplace equation is constructed in section 3. Its properties are discussed and the simplest physical applications are given. The multipole toroidal moments appear naturally in this basis and this removes the mystery of their origin. The conditions under which the periodical charge and current densities generate electromagnetic field strengths confined to the finite space region are found. The electromagnetic properties of TS with non-zero helicity, the influence of TS twisting on the Aharonov–Bohm scattering, and an alternative viewpoint on TS with non-zero helicity are discussed in section 4.

2. Main facts concerning elementary vector potentials and vector spherical harmonics

Consider the non-uniform wave equation for the scalar and vector potentials (VP)

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A=-\frac{4\pi}{c}j \qquad \left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi=-4\pi\rho \tag{2.1}$$

with charge and current densities periodically changing with time

$$\rho=\rho_0\exp(-\mathrm{i}\omega t) \qquad j=j_0\exp(-\mathrm{i}\omega t).$$

Clearly, A and Φ should be of the form

$$A=A_0\exp(-\mathrm{i}\omega t) \qquad \Phi=\Phi_0\exp(-\mathrm{i}\omega t). \tag{2.2}$$

Then

$$(\Delta+k^2)A_0=-\frac{4\pi}{c}j_0 \qquad (\Delta+k^2)\Phi_0=-4\pi\rho_0. \tag{2.3}$$

We assume that ρ and j are confined to the finite space region S. Outside it

$$\Phi_0=4\pi\mathrm{i}k\sum h_l Y_l^m q_l^m \tag{2.4}$$

$$A=\frac{1}{c}4\pi\mathrm{i}k\sum A_l^m(\tau)b_l^m(\tau). \tag{2.5}$$

Here

$$h_l\equiv h_l(kr) \qquad j_l\equiv j_l(kr)$$

are the spherical Bessel and Hankel functions, $Y_l^m \equiv Y_l^m(\theta, \varphi)$ are the usual spherical harmonics, and $q_l^m = \int j_l Y_l^{m*} \rho_0 \, dV$.

The $A_l^m(\tau)$ are the so called elementary vector potentials (EVP). The values of $\tau = E$, L and M in them correspond to the electric, longitudinal and magnetic multipoles. The $A_l^m(\tau)$ being the solutions of the Helmholtz equation are given by

$$A_l^m(M) = \frac{1}{\sqrt{l(l+1)}} L h_l Y_l^m \qquad A_l^m(L) = \frac{1}{k} \nabla h_l Y_l^m$$

$$A_l^m(E) = \frac{1}{\mathrm{i}k} \frac{1}{\sqrt{l(l+1)}} \mathrm{rot}(L h_l Y_l^m) \qquad L = -\mathrm{i}(r \times \nabla). \tag{2.6}$$

Another set of EVP $B_l^m(\tau)$ (non-singular at the origin) is obtained if we change the spherical Hankel functions h_l to the Bessel ones j_l. The multipole formfactors $b_l^m(\tau)$ occurring in (2.5) depend on the current distribution inside S

$$b_l^m(\tau) = \int B_l^m(\tau)^* \cdot j_0 \, dV. \tag{2.7}$$

The EVP have a number of nice properties. They are orthogonal on the sphere of arbitrary radius

$$\int A_l^m(\tau) \cdot A_{l'}^{m'}(\tau')^* \, d\Omega = \text{constant} \, \delta_{ll'} \, \delta_{mm'} \cdot \delta_{\tau\tau'}. \tag{2.8}$$

They are the eigenfunctions of the total angular momentum and its third projection

$$J^2 A_l^m(\tau) = l(l+1) A_l^m(\tau) \qquad J_z \cdot A_l^m(\tau) = m \cdot A_l^m(\tau)$$

(see [1] for the definition of J). The following differential relations between EVP are valid:

$$\mathrm{rot} A_l^m(M) = \mathrm{i}\kappa A_l^m(E) \qquad \mathrm{rot} A_l^m(E) = -\mathrm{i}\kappa A_l^m(M). \tag{2.9}$$

The same equations occur for $B_l^m(\tau)$. The EVP form a complete system. An arbitrary vector function can be developed over them. An alternative representation of VP is its expansion over the vector spherical harmonics (VSH)

$$A = \frac{1}{c} 4\pi \, \mathrm{i}\kappa \sum h_l(\kappa r) Y_{jl}^m(\theta, \varphi) \cdot J_{jl}^m. \tag{2.10}$$

Here $J_{jl}^m = \int j_l(kr) Y_{jl}^{m*} \cdot j_0 \, dV$. The VSH are defined as vectorially coupled quantities of the usual spherical harmonics and the unit spherical vectors $n_\mu (n_0 = n_z$, $n_{\pm 1} = \mp (n_x \pm \mathrm{i} n_y)/\sqrt{2})$

$$Y_{jl}^m = \sum C(1, \mu, l, m-\mu; jm) Y_l^{m-\mu} n_\mu. \tag{2.11}$$

Here $C(l_1 m_1 l_2 m_2; lm)$ are the usual Clebsch–Gordan coefficients. The VSH are orthonormal

$$\int Y_{jl}^m \cdot Y_{j'l'}^{m'*} \, d\Omega = \delta_{jj'} \delta_{ll'} \cdot \delta_{mm'}.$$

They are the eigenfunctions of the orbital and total angular moments squares and of the third projection of the latter:

$$L^2Y^m_{jl}=l(l+1)\,Y^m_{jl} \qquad J^2Y^m_{jl}=j(j+1)Y^m_{jl},\ J_zY^m_{jl}=mY^m_{jl}.$$

It is clear that $h_lY^m_{jl}$ and $j_lY^m_{jl}$ are the vector solutions of the Helmholtz equation. This suggests that EVP may be expressed in terms of the VSH [1]:

$$\begin{aligned}
A^m_l(M)&=-h_lY^m_{ll}\\
A^m_l(E)&=(\sqrt{l+1}\,h_{l-1}Y^m_{l,l-1}-\sqrt{l}\,h_{l+1}Y^m_{l,l+1})/\sqrt{2l+1}\\
A^m_l(L)&=(\sqrt{l+1}\,h_{l+1}Y^m_{l,l+1}+\sqrt{l}\,h_{l-1}Y^m_{l,l-1})/\sqrt{2l+1}.
\end{aligned}\tag{2.12}$$

The EVP $B^m_l(\tau)$ are obtained if one takes in these relations $j_l(kr)$ instead of $h_l(kr)$. The advantage of EVP over VSH is that EVP may be obtained by the action of the ∇ and L operators on the solutions of the scalar Helmholtz equation.

3. Vector solutions of the Laplace equation

It is our first goal to find an analogue of the EVP expansion for the Laplace equation. At first this problem seems to be almost trivial. In fact, as the Helmholtz equation (2.3) transforms in the $k\to 0$ limit into the Poisson one, one expects that it is enough to find the solution of the Helmholtz equation and then take the limit $k\to 0$. The following counterexample shows that the direct transition to the limit $k\to 0$ does not exhaust all the vector solutions of the Laplace equation.

Counterexample. Suppose we seek for the periodically changing with time charge and current densities confined to the finite space region S, and generating the electromagnetic strengths E, H vanishing outside S. To find conditions for this we use the curl operator on both sides of (2.5). Then

$$H=\frac{1}{c}4\pi k^2\sum[b^m_l(E)\cdot A^m_l(M)-b^m_l(M)\cdot A^m_l(E)].$$

As $A^m_l(E)$ and $A^m_l(M)$ are linear, independent and orthogonal, the conditions for the disappearance of H are

$$b^m_l(E)=b^m_l(M)=0.\tag{3.1}$$

The physical meaning of these equations is clarified in the appendix.
The corresponding VP is given by

$$A=\frac{4\pi i}{ck}\nabla\sum h_lY^m_l\int\nabla(j_lY^{m*}_l)\cdot j\,dV.\tag{3.2}$$

(The overall periodical factor $\exp(-i\omega t)$ is dropped in this and other evident cases); A and Φ satisfy the Lorentz gauge condition $\operatorname{div}A+\dot{\Phi}/c=0$. It is easy to check that the electric field $E=-\nabla\Phi-\frac{1}{c}\dot{A}$ also disappears outside S. Using the continuity equation $\operatorname{div}j+\dot{\rho}=0$ the VP may be transformed into

$$A=\frac{1}{ik}\nabla\cdot\Phi.\tag{3.3}$$

Consider the case when $kr \ll 1$. This means that the field observations are made at distances which are much smaller than the wavelength $\lambda = 2\pi/k$. On the other hand, we do not expand the overall time factor $\exp(-i\omega t)$. This permits us to observe the electromagnetic field at the given distance from the source at different times. Thus, we have

$$\Phi = \Phi_0 \exp(-i\omega t) \qquad A = \frac{1}{ik}\exp(-i\omega t)\nabla\Phi_0$$
$$\Phi_0 = 4\pi \sum \frac{1}{2l+1} r^{-l-1} Y_l^m \int r'^l Y_l^{m*} \rho_0(r')\, dV'. \tag{3.4}$$

To obtain the physical potentials one should take the real parts from

$$\Phi_{\text{phys}} = \Phi_0 \cdot \cos\omega t \qquad A_{\text{phys}} = -\frac{1}{k}\sin\omega t \nabla\Phi_0. \tag{3.5}$$

For small times ($\omega t \ll 1$) this reduces to

$$\Phi_{\text{phys}} = \Phi_0 \qquad A_{\text{phys}} = -ct\nabla\Phi_0. \tag{3.6}$$

We see that in the $k \to 0$ limit the solution (3.2) corresponds to the VP linearly growing with time. The particular solutions of such a kind have recently been discussed in [8]. For the pure current source ($\rho_0 = 0$) the solution (3.2) has a trivial limit $A = \Phi = 0$. We know, on the other hand, that static current distributions exist outside for which $H = E = 0$ but $A \neq 0$ (e.g. cylindrical and toroidal solenoids). This VP cannot be eliminated by the gauge transformation since there are closed paths along which $\oint A_l\, dl \neq 0$. This means that the non-trivial static solutions have been lost during the transition to the limit.

3.1. Transition to the longwavelength limit

We return to the initial equations (2.5), (2.10) and try to take this limit there. Equation (2.10) is transformed into

$$A = \frac{4\pi}{c} \sum \frac{1}{2l+1} r^{-l-1} \boldsymbol{Y}_{jl}^m j_{jl}^m \qquad j_{jl}^m = \int r^l \boldsymbol{Y}_{jl}^{m*} \cdot \boldsymbol{j}_0\, dV. \tag{3.7}$$

Clearly, $r^l \boldsymbol{Y}_{jl}^m$ and $r^{-l-1}\boldsymbol{Y}_{jl}^m$ are the vector solutions of the Laplace equation. They are eigenfunctions of L^2, J^2 and J_z. We are interested in those vector solutions which are expressible in a form similar to (2.6). However, we cannot form from $r^l\boldsymbol{Y}_{jl}^m$ or $r^{-l-1}\boldsymbol{Y}_{jl}^m$ linear combinations similar to (2.12) since the terms with different l have different dimensions and there is no constant (such as the wavenumber in the non-static case) to make them-dimensionless. Since EVP (2.6) have the form which we seek, it is natural to develop them in powers of k

$$\boldsymbol{A}_l^m(\tau) = k^{-l-2}[\boldsymbol{A}_{1l}^m(\tau) + k^2 \boldsymbol{A}_{2l}^m(\tau)]$$
$$\boldsymbol{B}_l^m(\tau) = k^{l-1}[\boldsymbol{B}_{1l}^m(\tau) + k^2 \boldsymbol{B}_{2l}^m(\tau)] \qquad \tau = L, E \tag{3.8}$$
$$\boldsymbol{A}_l^m(M) = k^{-l-1}\boldsymbol{A}_{1l}^m(M) \qquad \boldsymbol{B}_l^m(M) = k^l \boldsymbol{B}_{1l}^m(M).$$

The explicit values of the vector functions entering into the RHS of this equation are given in [15]. They are independent of k. The terms with higher powers of k do not contribute in the long wave-length limit and they are omitted in the development (3.8). The formfactors $b_l^m(\tau)$ entering into the definition (2.5) of VP may be also developed in powers of k

$$b_l^m(\tau)=k^{l-1}[b_{1l}^m(\tau)+k^2 b_{2l}^m(\tau)] \qquad \tau=L,E$$

$$b_l^m(M)=k^l b_{1l}^m(M) \qquad b_{1l}^m(\tau)=\int B_{1l}^{m*}(\tau)\cdot j_0\,\mathrm{d}V \tag{3.9}$$

$$b_{2l}^m(\tau)=\int B_{2l}^{m*}(\tau)\cdot j_0\,\mathrm{d}V.$$

It follows from (2.5), (3.8) and (3.9) that the contributions of the E and L multipoles taken separately diverge in the long wavelength limit like k^{-2}. On the other hand, the development (2.10) which is completely equivalent to (2.5) turns, in the same limit, into (3.7). No singularities arise during this transition. This means that singularities of the E and L multipoles in (2.5) compensate each other. In fact, the singular term appearing in (2.5) is given by

$$k^{-2}(b_{1l}^m(E)A_{1l}^m(E)+b_{1l}^m(L)\cdot A_{1l}^m(L)).$$

It is easy to check that this equation vanishes when the exact values of $A_{1l}^m(\tau)$ and $b_{1l}^m(\tau)$ are substituted into it. After these preliminaries we obtain the static limit of (2.5)

$$A=\frac{4\pi}{c}\sum\frac{1}{2l+1}\frac{1}{l(l+1)}C_l^m(M)\cdot d_l^m(M)$$

$$+\frac{2\pi}{c}\sum\frac{1}{4l^2-1}C_l^m(E)\cdot d_l^m(L)-\frac{2\pi}{c}\sum\frac{1}{(2l+1)(2l+3)}C_l^m(L)\cdot d_l^m(E). \tag{3.10}$$

Here

$$C_l^m(M)=(r\times\nabla)r^{-l-1}Y_l^m \qquad D_l^m(M)=(r\times\nabla)r^lY_l^m \tag{3.11}$$

$$C_l^m(E)=\left[\nabla-\frac{1}{l}\nabla\times(r\times\nabla)\right]r^{1-l}Y_l^m \qquad D_l^m(E)=\left[\nabla+\frac{1}{l+1}\nabla\times(r\times\nabla)\right]r^{l+2}Y_l^m$$

$$C_l^m(L1=\nabla r^{-1-l}Y_l^m \qquad D_l^m(L)=\nabla r^lY_l^m \qquad d_l^m(\tau)=\int D_l^{m*}(\tau)\cdot j_0\cdot\mathrm{d}V.$$

These are just expressions we need. The vector functions $C_l^m(\tau)$ and $D_l^m(\tau)$ are the vector solutions of the Laplace equation. This is not evident for $\tau=E$. In fact, the particular terms entering into the definitions of $C_l^m(E)$ and $D_l^m(E)$ do not satisfy the Laplace equation: only their linear combination does. In addition, $C_l^m(\tau)$ and $D_l^m(\tau)$ satisfy the following equations:

$$\operatorname{div}C_l^m(E)=-2\cdot(2l-1)r^{-l-1}Y_l^m \qquad \operatorname{div}D_l^m(E)=2\cdot(2l+3)r^lY_l^m$$

$$\operatorname{rot}C_l^m(E)=-\frac{2}{l}(2l-1)C_l^m(M) \qquad \operatorname{rot}D_l^m(E)=-\frac{2(2l+3)}{l+1}D_l^m(M)$$

$$\operatorname{rot}C_l^m(M)=lC_l^m(L) \qquad \operatorname{rot}D_l^m(M)=-(l+1)D_l^m(L)$$

$$\operatorname{div}C_l^m(M)=\operatorname{div}D_l^m(M)=\operatorname{div}C_l^m(L)=\operatorname{div}D_l^m(L)=\operatorname{rot}C_l^m(L)=\operatorname{rot}D_l^m(L)=0. \tag{3.12}$$

Like EVP they are orthogonal on the surface of the sphere. Taking into account the continuity equation ($\operatorname{div} j_0 = i\omega\rho_0$) we may transform $d_l^m(L)$ and $d_l^m(E)$ into $d_l^m(L) = -i\omega \int r^l Y_l^{m*}\rho_0 \, dV$ and

$$d_l^m(E) = \frac{1}{l+1}\int \nabla\times(r\times\nabla)r^{l+2}Y_l^{m*}\cdot j_0 \, dV - i\omega\int r^{l+2}Y_l^{m*}\rho_0 \, dV.$$

This means that in the limit $k\to 0$

$$d_l^m(L) = 0 \qquad d_l^m(E) = \frac{2\cdot(2l+3)}{l+1}\int r^l Y_l^{m*}(rj)\, dV.$$

As a result we have

$$A = -\frac{4\pi}{c}\sum\frac{1}{2l+1}\frac{1}{l(l+1)}r^{-l-1}\boldsymbol{C}_l^m(M)\cdot\int r^l Y_l^{m*}(\boldsymbol{r}\cdot \operatorname{rot} \boldsymbol{j})\, dV$$
$$-\frac{4\pi}{c}\nabla\sum\frac{1}{2l+1}\frac{1}{l+1}r^{-l-1}Y_l^m\cdot\int r^l Y_l^{m*}(\boldsymbol{r}\cdot\boldsymbol{j})\, dV. \tag{3.13}$$

It follows from this that H does not go beyond the space region S where $j\neq 0$ if J satisfies the condition

$$\boldsymbol{r}\cdot \operatorname{rot} \boldsymbol{j} = 0. \tag{3.14}$$

The same condition may be formulated on the magnetization language ($\boldsymbol{j} = c \operatorname{rot} \boldsymbol{M}$). It turns out that $\boldsymbol{H}$ disappears outside S, if it is filled by the substance with divergence-free ($\operatorname{div}\boldsymbol{M} = 0$) magnetization [16]. This fact is intuitively used by the experimentalists [17]).

3.2. *The toroidal multipole moments*

Now we turn again to (2.12) and similar ones for $\boldsymbol{B}_l^m(\tau)$. We develop both sides of these equations in powers of k and compare the coefficients at the same power of k. For $\tau = M$ we get from (2.12)

$$r^{-l-1}\boldsymbol{Y}_{ll}^m = \frac{i}{\sqrt{l(l+1)}}\boldsymbol{C}_l^m(M) \qquad r^l\boldsymbol{Y}_{ll}^m = \frac{i}{\sqrt{l(l+1)}}\boldsymbol{D}_l^m(M). \tag{3.15}$$

For $\tau = E, L$ we compare the coefficients at k^{-l-2} and k^{-1}

$$r^{-l-2}\boldsymbol{Y}_{l,l+1}^m = [(l+1)(2l+1)]^{-1/2}\boldsymbol{C}_l^m(L)$$
$$\operatorname{rot}(\boldsymbol{r}\times\nabla)r^{1-l}Y_l^m = l\cdot(2l-1)\sqrt{\frac{l+1}{2l+1}}\,r^{-l}\left(\boldsymbol{Y}_{l,l+1}^m - \sqrt{\frac{l+1}{l}}\frac{1}{l-\frac{1}{2}}\boldsymbol{Y}_{l,l-1}^m\right)$$
$$\nabla r^{1-l}Y_l^m = \sqrt{\frac{l+1}{2l+1}}\,r^{-l}\left(\boldsymbol{Y}_{l,l+1}^m + \sqrt{\frac{l}{2l+1}}\frac{1}{l-\frac{1}{2}}\boldsymbol{Y}_{l,l-1}^m\right)(2l-1). \tag{3.16}$$

The two last equations may be reversed

$$r^{-l}\boldsymbol{Y}_{l,l-1}^m = \frac{1}{2}\sqrt{\frac{l}{2l+1}}\left[\nabla - \frac{1}{l}\operatorname{rot}(\boldsymbol{r}\times\nabla)\right]r^{1-l}Y_l^m$$
$$r^{-l}\boldsymbol{Y}_{l,l+1}^m = \frac{1}{2l-1}\sqrt{\frac{l+1}{2l+1}}\left[\nabla + \frac{1}{l+1}\operatorname{rot}(\boldsymbol{r}\times\nabla)\right]r^{1-l}Y_l^m. \tag{3.17}$$

For $\tau = E, L$ we equalize the coefficients at k^{l-1} and k^{l+1} in the development of $B_l^m(\tau)$:

$$r^l Y_{l,l+1}^m = [(l+1)(2l+3)]^{-1/2} D_{l+1}^m(L)$$

$$\mathrm{rot}(\boldsymbol{r}\times\nabla) r^{l+2} Y_l^m = -(l+1)(2l+3)\sqrt{\frac{l}{2l+1}}\, r^{l+1}\left(Y_{l,l-1}^m + \sqrt{\frac{l}{l+1}}\frac{1}{l+\frac{3}{2}} Y_{l,l+1}^m\right)$$

$$\nabla r^{l+2} Y_l^m = (2l+3)\sqrt{\frac{l}{2l+1}}\, r^{l+1}\cdot\left(Y_{l,l-1}^m - \sqrt{\frac{l+1}{l}}\frac{l}{l+\frac{3}{2}} Y_{l,l+1}^m\right). \tag{3.18}$$

Reversing the two last equations we obtain

$$r^{l+1} Y_{l,l+1}^m = -\frac{1}{2}\sqrt{\frac{l+1}{2l+1}}\left[\nabla + \frac{1}{l+1}\mathrm{rot}(\boldsymbol{r}\times\nabla)\right] r^{l+2} Y_l^m$$

$$r^{l+1} Y_{l,l-1}^m = \frac{1}{2l+3}\sqrt{\frac{l}{2l+1}}\left[\nabla - \frac{1}{l}\mathrm{rot}(\boldsymbol{r}\times\nabla)\right] r^{l+2} Y_l^m. \tag{3.19}$$

The first part of (3.18), when folded with the current density and integrated over the volume, gives

$$\int \mathrm{rot}(\boldsymbol{r}\times\nabla) r^{l+2} Y_l^{m*} \boldsymbol{j}_0\, \mathrm{d}V$$

$$= -(l+1)(2l-3)\sqrt{\frac{l}{2l+1}}\int r^{l+1}\left(Y_{l,l-1}^{m*}\sqrt{\frac{l}{l+1}}\frac{1}{l+\frac{3}{2}} Y_{l,l+1}^{m*}\right)\boldsymbol{j}_0\, \mathrm{d}V.$$

The integral in the RHS coincides with the so-called toroidal moment [7]. Then the integral in the LHS may be viewed as its alternative representation. If $\mathrm{div}\,\boldsymbol{j} = 0$ then the LHS is simplified:

$$2\cdot(2l+3)\int r^l Y_l^{m*}(\boldsymbol{r}\cdot\boldsymbol{j}_0)\, \mathrm{d}V.$$

Thus, a toroidal moment arises naturally as the coefficient at k^{l+1} in the development of electric formfactors. This removes the mystery of their origin. It follows from (3.15)–(3.19) that the expansion (3.10) is completely equivalent to the VSH one. The novelty is that we succeeded in presenting them in the differential form. From the vector identity

$$\mathrm{rot}(\boldsymbol{r}\times\nabla) r^\alpha Y_l^m = -(\alpha+1)\nabla r^\alpha Y_l^m + (\alpha - l)(\alpha + l + 1)\boldsymbol{r} r^{\alpha-2} Y_l^m$$

we obtain for $\alpha = l$ and $\alpha = -l-1$

$$\begin{aligned} &\mathrm{rot}(\boldsymbol{r}\times\nabla) r^l Y_l^m = -(l-1)\nabla r^l Y_l^m \\ &\mathrm{rot}(\boldsymbol{r}\times\nabla) r^{-l-1} Y_l^m = -l\nabla r^{-l-1} Y_l^m. \end{aligned} \tag{3.20}$$

It follows from this that there are vector functions that can be simultaneously presented as a curl and a gradient.

Separate pieces of the VP expansion corresponding to the particular choices of the current density have been used elsewhere [6, 18]. The expansion (3.10) permits one to solve the Poisson equation for an arbitrary stationary current.

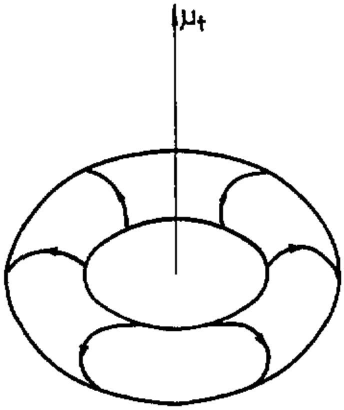

Figure 1. Poloidal current on the torus surface and the associated dipole toroidal moment.

Concrete example. As an example illustrating the usefulness of the relations obtained consider the toroidal solenoid

$$(\rho-d)^2+z^2=R^2 \tag{3.21}$$

with the poloidal current in its winding (figure 1). It is convenient to introduce the coordinates $\tilde{R}$, ψ: $\rho=d+\tilde{R}\cos\psi, z=\tilde{R}\cdot\sin\psi$. Then, the value $\tilde{R}=R$ corresponds to the solenoid (3.21). The poloidal current is given by

$$j_0=-\frac{gc}{4\pi}\frac{\delta(\tilde{R}-R)}{d+R\cos\psi}\boldsymbol{n}_\psi .$$

Here $g=2NI/c$, I is the current in a particular turn, N is the number of turns, $\boldsymbol{n}_\psi$ is the unit vector defining the current direction in a particular turn ($\boldsymbol{n}_\psi=\boldsymbol{n}_z\cos\psi(\boldsymbol{n}_x\cos\varphi+\boldsymbol{n}_y\sin\varphi)\sin\psi$). It is easy to see that rot J has only the φ component and, thus, the condition (3.14) is satisfied. As

$$rj=\frac{gcd}{4\pi}\frac{\delta(\tilde{R}-R)}{d+R\cos\psi}\sin\psi$$

only the $m=0$ component and odd values of l contribute to VP. Putting $l=2n+1$ we obtain

$$A=-\frac{1}{4}gdR\nabla\sum\frac{1}{n+1}r^{-2n-2}\cdot P_{2n+1}(\cos\theta)\cdot\int\rho^{2n+1}P_{2n+1}\left(\frac{R\sin\psi}{\rho}\right)\sin\psi\,\mathrm{d}\psi. \tag{3.22}$$

Here $\rho^2=d^2+R^2+2dR\cos\psi$ and P_n is the Legendre polynomial. For the infinitely thin ($R\ll d$) solenoid this reduces to

$$A=-\frac{1}{4}\pi gR^2\nabla\sum(-1)^n\frac{(2n+1)!!}{2^n\cdot(n+1)!}\frac{d^{2n+1}}{r^{2n+2}}P_{2n+1}(\cos\theta). \tag{3.23}$$

These equations are valid outside the sphere of radius $d+R$. Inside the sphere of radius $d-R$

$$A=\frac{1}{2}gRd\nabla\sum\frac{1}{2n+1}r^{2n+1}P_{2n+2}\int\frac{1}{\rho^{2n+2}}P_{2n+1}\left(\frac{R\sin\psi}{\rho}\right)\sin\psi\,\mathrm{d}\psi \tag{3.24}$$

for the finite TS and

$$A=\frac{1}{2}\pi g R^2\nabla\sum(-1)^n\frac{(2n-1)!!}{2^n n!}\frac{r^{2n+1}}{d^{2n+2}}P_{2n+1}(\cos\theta) \tag{3.25}$$

for the infinitely thin one. We observe that VP is equal to the gradient of some function χ. This function is different for $r>d+R$ and $r<d-R$. This merely reflects the fact that the VP cannot be removed by the guage transformation (as $\oint A_l \mathrm{d}l\neq 0$ for the paths passing through the TS hole). The whole derivation of the VP expansion for TS has taken only half a page. This is due to the use of (3.10). In contrast, one may compare this with a many-page derivation in [19].

3.3 Non-static charge–current configuration

We see that the expansion (3.10) includes the description of the usual magnetic solenoids. On the other hand (3.2), which defines the most general non-static VP with vanishing field strengths, becomes trivial in the $k\to 0$ limit. Here, it follows that there are no non-static charge–current configurations corresponding to the vanishing field strengths in the space surrounding them and reducing to the usual magnetic solenoids in the $k\to 0$ limit.

The charge–current distributions confined inside S and satisfying the conditions (3.1) do not radiate because $E=H=0$ outside S. On the other hand, it is possible to construct non-radiating systems with E, $H\neq 0$ outside S. This happens if the Poynting vector

$$P=\frac{1}{4\pi c}(E\times H)$$

decreases faster than r^{-2}. Consider the explicit expressions for the electromagnetic potentials

$$\Phi_0=\int G_k\rho_0(r')\,\mathrm{d}V' \qquad A_0=\frac{1}{c}\int G_k j_0\,\mathrm{d}V'$$

$$G_k=\exp(\mathrm{i}k|r-r'|)/|r-r'|.$$

At large distance one has

$$\Phi_0\approx\Phi'=\frac{1}{r}\exp(\mathrm{i}kr)\int\exp(-\mathrm{i}kn_r r')\rho_0(r')\,\mathrm{d}V' \qquad n_r=\frac{\mathrm{r}}{r}$$

$$A_0\approx A'=\frac{1}{cr}\exp(\mathrm{i}kr)\int\exp(-\mathrm{i}kn_r\cdot r')j_0(r')\,\mathrm{d}V'$$

$$E\approx E'=-\mathrm{i}kn_r\Phi'+\mathrm{i}kA' \qquad H=H'=\mathrm{i}k(n_r\times A'). \tag{3.26}$$

The terms of the order r^{-2} and higher are omitted since they do not contribute to the energy flux. The radial component of the Poynting vector equals

$$S_r=\frac{1}{4\pi c}(n_r\cdot(E'\times H'))=\frac{k^2}{4\pi c}(n_r(A'\times(A'\times n_r)))=\frac{1}{4\pi c}k^2(|A'|^2-|A_r'|^2)$$

$$=\frac{1}{4\pi c}k^2(|A_\theta'|^2+|A_\varphi'|^2).$$

It follows from this that the energy flux into the surrounding space vanishes if $A'_\theta = A'_\varphi = 0$. In a slightly different form these conditions have been obtained in [20], the particular realizations of such non-radiating systems may be found in [20, 21].

4. Magnetic solenoids with non-zero helicity

4.1 Cylindrical solenoids

We consider first the cylindrical solenoid C of radius R. Let the current $\boldsymbol{j} = j \cdot \boldsymbol{n}_\varphi \cdot \delta(\rho - R)$ flow on its surface. The corresponding VP is $\boldsymbol{A} = A \cdot \boldsymbol{n}_\Phi$ where $A = \Phi/2\pi\rho$ outside C and $\Phi\rho/2\pi R^2$ inside it. The magnetic field differs from zero only inside C: $\boldsymbol{H} = \boldsymbol{n}_z \Phi/\pi R^2$. Here Φ is the magnetic flux inside C: $\Phi = \int\int H_z \rho \, d\rho \, d\varphi = 4\pi^2 R^2 j$. In the treated case the magnetic field and VP are mutually orthogonal, so

$$S = \int \boldsymbol{A} \cdot \boldsymbol{H} \, dV = 0. \tag{4.1}$$

The quantity S is called helicity [12 − 14]. Thus, the usual cylindrical solenoid has zero helicity. Instead of the current J one may equally use the magnetization M: $\boldsymbol{j} = c \, \text{rot} \, \boldsymbol{M}$. For the treated case

$$\boldsymbol{M} = M \cdot \theta(R - \rho) \cdot \boldsymbol{n}_z \qquad M = \pi R^2 j.$$

It is convenient to forget about the initial current and treat the solenoid as a cylinder uniformly magnetized along its symmetry axis. Let the magnetization M have φ component (in addition to the existing Z one):

$$\boldsymbol{M} = M \cdot \theta(R - \rho)(\boldsymbol{n}_z \cdot \cos\alpha + \boldsymbol{n}_\varphi \cdot \sin\alpha). \tag{4.2}$$

This means that magnetic lines twist around the co-axial cylindrical surfaces which are confined inside the initial cylindrical solenoid C. The non-vanishing components of VP and magnetic induction are

$$A_z = 4\pi(R - \rho)\sin\alpha \cdot M \qquad A_\varphi = 2\pi M \rho \cos\alpha$$

$$B_\varphi = 4\pi M \sin\alpha \qquad B_z = 4\pi M \cos\alpha$$

inside the cylinder and

$$A_z = 0 \qquad A_\varphi = 2\pi R^2 M \cos\alpha/\rho \qquad \boldsymbol{B} = 0$$

outside it. As a result, the helicity per unit of the cylinder length equals

$$S = \frac{16}{3}\pi^3 M^2 \sin 2\alpha \, R^3.$$

What is important is that the cylindrical solenoid with non-zero helicity thus obtained consists exactly of the same magnetic lines as the original one (with zero helicity). Only their direction has been changed. The part Φ' of the total magnetic flux Φ which threads the Z = constant plane now equals $\Phi' = \Phi \cdot \cos\alpha$. Particularly, $\Phi' = 0$ for $\alpha = \pi/2$, i.e. when all magnetic lines lie in the Z = constant plane. In this case the cylindrical solenoid degenerates into the linear chain of toroidal dipole moments outside which the VP disappears. For $0 < \alpha < \pi/2$ the cylindrical solenoid may be viewed as the superposition of the dipole and toroidal moments distributed along the linear chains parallel to the Z axis.

4.2. Digression on toroidal moments

Likewise the circular current carries the magnetic dipole moment directed normally to the current plane, the poloidal current flowing on the torus suface (figure 1) carries the anapole [22] or toroidal [7] dipole magnetic moment μ_t (TDM, for short). It is directed along the torus symmetry axis. The TDM is also generated by the closed chain of magnetic dipoles. Its interaction with an external magnetic field up to a constant is given by $U \sim \mu_t \cdot \text{rot}\, H$. As outside the current sources $\text{rot}\, H = \dot{E}/c$ so

$$U \sim \frac{1}{c} \dot{E} \cdot \mu_t.$$

The existence of such an interaction was confirmed experimentally [23]. From this consideration it follows that a 'toroidal compass' measuring rot H, but not H itself, can be constructed. Its simplest realization is the ferromagnetic ring either installed on the platform having the rotational freedoms of motion (likewise the rotating disc in a usual gyroscope) or immersed into the imponderability (in the artificial satellite, in the insulator liquid). The axis of this ferromagnetic ring tends to be oriented along rot H. To obtain such a magnetic field, it is enough to apply the time-dependent voltage to the plates of a usual capacitor. The time variation of the electric field E (which itself is normal to the capacitor's plates) produces rot $H (= \dot{E}/c)$ having the same direction as E. Thus, the axis of the toroidal compass placed inside the capacitor tends to be oriented normally to the capacitor's plates. A more sensitive toroidal compass having a smaller weight is obtained if we use the hollow torus with a poloidal winding on its surface. The torque exhibited by this compass is proportional to the number of turns in its winding and to the current strength in each particular turn. Special precautions should be made to get rid of the current toroidal component. A usual compass (magnetic needle) measures both potential and solenoidal components of H. To separate their contributions, one should measure the magnetic field in the finite region of space. In contrast with this, the use of a toroidal compass permits one to detect the existence of a non-potential component of the magnetic field by performing only one measurement. The toroidal moments of higher multipolarities can be constructed [7, 15] from TDM (likewise, the TDM itself is constructed from the usual magnetic dipoles). They interact with higher derivatives of the magnetic field. The VP generated by them decreases at large distances more rapidly than the VP of TDM (which falls as r^{-3}). Outside the closed chain (or the infinite linear one) consisting of the usual magnetic dipoles, magnetic strength vanishes, but the VP differs from zero. Outside the closed chain (or the infinite linear one) composed of TDM both the magnetic strength and VP are equal to zero [10, 15]. Thus, this chain has a more pronounced self-screening (or 'black box') property than the chain composed of the usual magnetic dipoles.

Let us study the electron scattering on the twisted cylindrical solenoid (by this we mean that the magnetization has the form (4.2), i.e. magnetic lines are twisted around the Z axis). The scattering cross-section is determined by the value of the integral $\oint A_l \, dl$ taken along the contour encircling the solenoid. It is equal to the part of the total magnetic field which threads the Z = constant plane, i.e. to $\Phi \cdot \cos \alpha$. This means that scattering cross-section will be a periodical function of the twisting angle α. For $\alpha = \pi/2$ there is no VP outside the solenoid and, therefore, no scattering on it, i.e. the linear shielded chain of toroidal moments presents some kind of 'black box'.

Consider now two cylindrical samples manufactured of the same magnetic substance. We twist one of these samples (i.e. deform its magnetic lines according to (4.2)) and scatter the electrons on both of them. As the scattering cross-section depends on the twisting angle α, it is possible in principle to determine (except for the special combinations of Φ and α) which of the samples is twisted and which is not.

4.3. *Toroidal solenoids*

We turn now to the TS. It may be viewed as the set of magnetized filaments filling the torus T $(\rho-d)^2+z^2=R^2$. The magnetization and induction being expressed in the toroidal coordinates

$$\rho=\frac{a\sinh\mu}{\cosh\mu-\cos\theta} \qquad z=\frac{a\sin\theta}{\cosh\mu-\cos\theta}$$

(here μ defines the particular torus while θ and φ run over its surface) are equal to

$$\boldsymbol{M}=M\cdot\boldsymbol{n}_\varphi \qquad \boldsymbol{B}=4\pi\boldsymbol{M} \qquad M=M_0\frac{\cosh\mu-\cos\theta}{\sinh\mu}\cdot\theta(\mu-\mu_0)$$
$$(d=a\cdot\coth\mu_0, R=a/\sinh\mu_0). \tag{4.3}$$

The values $\mu>\mu_0$ and $\mu<\mu_0$ correspond to the points lying inside and outside the torus T, respectively. The constant M_0 may be also expressed through the magnetic flux $M_0=\Phi[(\coth\mu_0-1)8\pi^2a^2]^{-1}$. For the infinitely thin TS and for Φ fixed, M_0 tends to infinity: $M_0\sim\Phi\cdot\exp(2\mu_0)/16\pi^2a^2$. The VP of TS has been obtained in [24] and its properties have been discussed in [25]. The non-vanishing toroidal components of VP are A_μ and A_θ. It follows from this that TS with magnetization (4.3) possesses zero helicity. Let the magnetization $\boldsymbol{M}$ have the θ component (in addition to the existing φ one)

$$\boldsymbol{M}=M(\boldsymbol{n}_\varphi\cdot\cos\alpha+\boldsymbol{n}_\theta\cdot\sin\alpha) \qquad \boldsymbol{B}=4\pi\boldsymbol{M}. \tag{4.4}$$

This means again that the magnetization lines are twisted along the toroidal surfaces $(\mu_0<\mu<\infty)$ which are completely inside the torus $T(\mu=\mu_0)$. The θ component of $\boldsymbol{M}$ generates the φ component of VP which is different from zero only inside T:

$$A_\varphi=-8\pi M_0\cdot\sin\alpha\frac{\cosh\mu-\cos\theta}{\sinh\mu}\cdot\tan^{-1}\frac{\sin\theta\cdot\sinh\frac{\mu-\mu_0}{2}}{\cosh\frac{\mu+\mu_0}{2}-\cos\theta\cosh\frac{\mu-\mu_0}{2}}. \tag{4.5}$$

Such a solenoid may be viewed as a superposition of the magnetic dipole and toroidal dipole moments distributed along the closed circular chains lying in the $Z=$constant planes and imbedded into the torus T (for details see again in [10, 15]). This superposition reduces to the closed chain consisting of the usual magnetic dipoles for $\alpha=0$ and of the TDM for $\alpha=\pi/2$.

The μ and θ components of the VP are obtained from those found in [24] by multiplying them by the factor $\cos\alpha$. It follows from this that non-zero helicity corresponds to the magnetization (4.4). Since the VP components are rather complicated for the finite TS we limit ourselves to the infinitely thin one ($R\ll d$ or $\mu_0\gg1$). In this case, the following components of VP and the magnetic induction differ from zero inside the TS

$$A_\theta = 4\pi a M_0[\exp(-\mu) - \mu_0 \cdot \cos\theta \cdot \exp(-\mu_0)] \cos\alpha$$
$$A_\varphi = -8\pi a M_0 \cdot \exp(-\mu_0) \cdot \sin\alpha$$
$$B_\varphi = 4M_0\pi \cos\alpha \qquad B_\theta = 4\pi M_0 \sin\alpha. \tag{4.6}$$

As a result, we obtain for the helicity:

$$S = \int \boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}V = \tfrac{1}{3}\, 32\pi^4 a^4 M_0^2 \sin 2\alpha \cdot \exp(-3\mu_0). \tag{4.7}$$

The question arises: is it possible to get information on the helicity by performing experiments outside TS (which may be surrounded by the impenetrable torus)? We note that the θ component of magnetization does not contribute to the VP outside T. The wavefunction [26] describing the scattering of the charged particles on the impenetrable toroidal solenoid depends on its geometrical dimensions (d, R) and on the part Φ' of the total magnetic flux Φ which threads the φ = constant plane of the solenoid (it is just this part of Φ that generates the non-zero VP outside TS):

$$\psi = \exp(\mathrm{i}kz) + \psi_s$$
$$\psi_s = \mathrm{i}\frac{1+\cos\theta_s}{2} \exp\left(\mathrm{i}k\frac{d^2+R^2}{2r}\right) [\exp(\mathrm{i}w) \cdot W_1 - \exp(\mathrm{i}\gamma - \mathrm{i}\omega) \cdot W_1].$$

Here θ_s and r are the scattering angle and distance from the solenoid to the observation point, $\omega = KdR/r$, $\gamma = e\Phi \cos\alpha/\hbar e$, W_1 and W_2 are the linear combinations of the Lommel functions of two variables:

$$W_{1,2} = U_1\left[\frac{k(d\pm R)^2}{r}, k(d\pm R)\sin\theta\right] - \mathrm{i}\, U_2\left[\frac{k(d\pm R)^2}{r}, k(d\pm R)\sin\theta\right].$$

It follows from this that the intensity $I = |\psi|^2$ is a periodical function of the angle α. Physically, this means that the change of the twisting angle α does not change the total magnetic flux but changes its component normal to the φ = constant plane. As an integral $\oint A_l\, \mathrm{d}l$ (which defines the AB scattering amplitude), taken along the contour passing through the TS hole, is equal to this component of flux, the dependence on the twisting angle α becomes evident. Consider two toroidal samples fabricated of the same magnetic substance. By cutting one of them, twisting it and reconnecting again we obtain the toroidal sample with non-zero helicity [14]. The same considerations as for the cyindrical solenoid show that twisted and non-twisted toroidal samples should exhibit different quantum-mechanical scattering.

4.4. *An alternative interpretation of the solenoids with non-zero helicity*

The usual explanation of non-zero helicity for TS composed of twisted magnetic lines proceeds as follows. Consider the torus with two magnetic lines winding on its surface. It is easy to see that after removing the torus we cannot decouple magnetic lines without cutting at least one of them. This suggests that these two magnetic lines form a non-trivial topological configuration. In fact, they correspond to non-zero helicity [14]. This remains valid for the continuum of the twisted magnetic lines filling the interior of the torus. The considered model of TS with non-zero helicity is of more prosaic, less topological nature. It treats such a solenoid as the superposition of closed chains consisting of the usual magnetic dipoles and TDM. The magnetic flux Φ and helicity S are the simplest representatives of the topological invariants, characterizing the structure of the static magnetic field and remaining the same for the arbitrary continuous deformation of the solenoids. There exist topological invariants different from Φ and S [27], which describe more subtle features of the static magnetic field. The latter can also be described by using higher order toroidal moments. This parallelism suggests that there are two alternative (or equivalent?) languages for the description of static magnetic field.

7. Conclusions

We briefly summarize the main results obtained:

(1) The vector solutions of the Laplace equation are presented in a convenient differential form. This makes easier the solution of various magnetostatic problems. The simplest applications of these solutions are given.

(2) The conditions are found under which the periodically changing with time charge and current densities do not radiate.

(3) The electromagnetic properties of the toroidal solenoids with non-zero helicity, the influence of the latter on the Aharonov–Bohm scattering and an alternative viewpoint on the toroidal solenoid with non-zero helicity are discussed.

Acknowledgements

This consideration arose from numerous discussions with Professor Ya A Smorodinsky whose passing away was so untimely.

Appendix

The explicit form of equations (3.1) is

$$\int j_l Y_l^{m*} \cdot (\boldsymbol{r} \cdot \mathrm{rot}\, \boldsymbol{j})\, \mathrm{d}V = 0$$
$$\int j_l Y_l^{m*} \cdot (\boldsymbol{r} \cdot \mathrm{rot}\,\mathrm{rot}\, \boldsymbol{j})\, \mathrm{d}V = 0. \qquad \text{(A.1)}$$

An arbitrary vector function and, particularly, the current density can be represented in the form (Helmholtz parametrization [28])

$$\boldsymbol{j} = \nabla\Psi_1 + \mathrm{rot}(\boldsymbol{r}\Psi_2) + \mathrm{rot}(\boldsymbol{r} \times \nabla\Psi_3)$$

or, by rearranging the terms,

$$\boldsymbol{j} = \nabla\Psi'_j + (\boldsymbol{r} \times \nabla)\Psi'_2 + \boldsymbol{r}\Psi'_3. \qquad \text{(A.2)}$$

To find Ψ_i' one applies to $\boldsymbol{j}$ the div and rot operators

$$(\boldsymbol{r}\cdot\boldsymbol{j})=r\frac{\mathrm{d}}{\mathrm{d}r}\Psi_1'+r^2\Psi_3'$$

$$r^2\,\mathrm{div}\,\boldsymbol{j}=(\boldsymbol{r}\times\nabla)^2\Psi_1'+\frac{\mathrm{d}}{\mathrm{d}r}(r(\boldsymbol{r}\cdot\boldsymbol{j})) \tag{A.3}$$

$$\boldsymbol{r}\cdot\mathrm{rot}\,\boldsymbol{j}=(\boldsymbol{r}\times\nabla)^2\Psi_2'$$

$$\boldsymbol{r}\cdot\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{j}=-(\boldsymbol{r}\times\nabla)^2\Psi_3'.$$

As a result, the following equations are obtained for Ψ_i'

$$(\boldsymbol{r}\times\nabla)^2\Psi_1'=r^2\,\mathrm{div}\,\boldsymbol{j}-\frac{\mathrm{d}}{\mathrm{d}r}(r(\boldsymbol{r}\cdot\boldsymbol{j}))$$

$$(\boldsymbol{r}\times\nabla)^2\Psi_2'=\boldsymbol{r}\cdot\mathrm{rot}\,\boldsymbol{j} \tag{A.4}$$

$$(\boldsymbol{r}\times\nabla)^2\Psi_3'=-\boldsymbol{r}\cdot\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{j}.$$

Consider the equation

$$(\boldsymbol{r}\times\nabla)^2\Psi=f. \tag{A.5}$$

Its Green function

$$G(\boldsymbol{n},\boldsymbol{n}')=-\sum_{lm,\,l\geq 1}\frac{1}{l(l+1)}Y_l^m(\boldsymbol{n})Y_l^{m*}(\boldsymbol{n}') \qquad \boldsymbol{n}=(\theta,\varphi)$$

satisfies the equation

$$(\boldsymbol{r}\times\nabla)^2G(\boldsymbol{n},\boldsymbol{n}')=\frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\varphi-\varphi')-\frac{1}{4\pi}.$$

Then,

$$\Psi=\int G(\boldsymbol{n},\boldsymbol{n}')f(\boldsymbol{r}')\,\mathrm{d}\Omega'. \tag{A.6}$$

The functions Ψ_i' are easily found if one substitutes the right-hand sides of (A.4) instead of $f(\boldsymbol{r})$. By taking account of (A.4), the non-radiation conditions (A.1) take the form

$$\int j_l Y_l^{m*}(\boldsymbol{r}\times\nabla)^2\Psi_2'\,\mathrm{d}V=0$$

$$\int j_l Y_l^{m*}(\boldsymbol{r}\times\nabla)^2\Psi_3'\,\mathrm{d}V=0. \tag{A.7}$$

Or by integrating by parts

$$l(l+1)\int j_l Y_l^{m*}\Psi_2'\,\mathrm{d}V=0$$

$$l(l+1)\int j_l Y_l^{m*}\Psi_3'\,\mathrm{d}V=0. \tag{A.8}$$

In addition to the trivial solutions $\boldsymbol{r}\cdot\mathrm{rot}\,\boldsymbol{j}=0$, $\boldsymbol{r}\cdot\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{j}=0$ (or, which is the same,

$(\boldsymbol{r}\times\nabla)^2\Psi_2'=0$, $(\boldsymbol{r}\times\nabla)^2\Psi_3'=0$) equations (A.1), (A.2), (A.8) have solutions corresponsing to the disappearance of integrals (but not the integrands) occurring there. This may take place for the definite values of the wavenumber k. The particular examples of non-radiating systems studied in [20, 21] certainly meet the conditions (A.1), (A.7), (A.8). In deriving these equations it has been implicitly assumed that the space region lying outside S (where $\rho, \boldsymbol{j}$) is a simply connected one. Let this region be a multi-connected one. For example, let it coincide with the torus $(\rho-d)^2+z^2=R^2$. Then, for $r>d+R$ the non-radiation conditions are again (A.2), (A.7), (A.8). For $r<d-R$ the spherical Bessel functions should be replaced by the Neumann ones:

$$\int n_l Y_l^{m*}(\boldsymbol{r}\times\nabla)^2\Psi_2'\,\mathrm{d}V=0$$
$$\int n_l Y_l^{m*}(\boldsymbol{r}\times\nabla)^2\Psi_3'\,\mathrm{d}V=0. \tag{A.9}$$

Or by integrating by parts

$$l(l+1)\int n_l Y_l^{m*}\Psi_2'\,\mathrm{d}V=0$$
$$l(l+1)\int n_l Y_l^{m*}\Psi_3'\,\mathrm{d}V=0. \tag{A.10}$$

It is very difficult to find non-static charge-current distributions for which equations (A.7), (A.8) and (A.9), (A.10) are simultaneously fulfilled.

Up to now, we have dealt with pure volume non-radiating charge-current distributions. The situation changes if we take into account the surface (in addition to the volume) distributions. According to the generalized Green theorem (see, e.g., [29] or [30]) an arbitrary charge-current distribution (static or periodically changing with time) enclosed inside the volume V can be simulated by the electric charges, dipoles and currents properly distributed over the surface enclosing this volume. It immediately follows from this [8] that it is possible to find surface distributions of charges and currents which completely compensate the electromagnetic field produced by the volume charge-current distribution (in fact, this is a usual electromagnetic shielding widely used by the experimentalists). The answer to the following question remains unclear to us: 'Is it possible to find surface charge-current distributions that completely compensate electromagnetic field strengths but not the potentials outside that surface?'

References

[1] Rose M E 1955 *Multipole Fields* (New York: Wiley)

[2] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vol 2 (New York: McGraw Hill) Ch 13

Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley) Ch 16

[3] Belkic D 1992 *Physica Scripta* **45** 9

[4] Rennert P 1990 *Ann. der Physik* **47** 27

[5] Blatt J M and Weisskopf V F 1952 *Theoretical Nuclear Physics* (New York: Wiley) Appendix B

[6] Boston E R and Sanders P G H 1990 *J. Phys. B: At. Mol. Phys.* **23** 2663

[7] Dubovik V M and Tugushev V V 1990 *Phys. Rep.* **187** 145

[8] Miller M A 1984 *Usp. Fiz. Nauk* **142** 147; 1986 *Izv. Vys. Uch. Zav., Radiofiz.* **29** 391
[9] Petuchov V R 1991 ITEP *Preprints* 105-91, 106-91
[10] Afanasiev G N 1993 *J. Phys. A: Math. Gen.* **26** 731
[11] Ranada A F 1992 *J. Phys. A: Math. Gen.* **25** 1621
[12] Ranada A F 1992 *Eur. J. Phys.* **13** 79
[13] Moffat H K 1990 *Nature* **347** 367
[14] Pfister H and Gekelman W 1991 *Am. J. Phys.* **59** 497
[15] Afanasiev G N 1993 *Fiz. Elm. Chastits At. Yadra* **24** 512
[16] Afanasiev G N, Dubovik V M and Misicu S 1993 *J. Phys. A: Math. Gen.* **26** 3279
[17] Tonomura A 1992 *Adv. Phys.* **41** 59
[18] Datta S 1984 *Eur. J. Phys.* **1** 243
[19] Afanasiev G N and Dubovik V M 1992 *J. Phys. A: Math. Gen.* **25** 4869
[20] Meyer-Vernet N 1989 *Am. J. Phys.* **57** 1084
Goedecke G 1964 *Phys. Rev.* B **135** 281
Pearle P 1977 *Found. Phys.* **7** 931
[21] Abbott T A and Griffiths D J 1985 *Am. J. Phys.* **53** 1203
Bohm D and Weinstein M 1948 *Phys. Rev.* **74** 1789
[22] Zeldovich Ya B 1958 *Sov. Phys.-JETP* **6** 1184
[23] Tolstoy N A and Spartakov A A 1990 *Zh. Eksp. Teor. Fiz. Pis. Red.* **51** 796
[24] Afanasiev G N 1987 *J. Comput. Phys.* **69** 196
[25] Afanasiev G N 1990 *J. Phys. A: Math. Gen.* **23** 5755
[26] Afanasiev G N and Shilov V M 1993 *J. Phys. A: Math. Gen.* **26** 743
[27] Berger M A 1990 *J. Phys. A: Math. Gen.* **23** 2787
[28] Elsasser W M 1946 *Phys. Rev.* **69** 106
[29] Stratton J A 1951 *Electromagnetic Theory* (New York: McGraw-Hill) ch 3, 4, 8
[30] Courant R and Hilbert D 1962 *Methods of Mathematical Physics* vol 2 (New York: Interscience) ch 5